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LETTER TO THE EDITOR

The notion of a strange fractal phenomenon—disorder induced walk

R B Pandey

Cavendish Laboratory, Cambridge CB3 0HE, UK

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Abstract. It is argued that a random system coupled with a walk may cause an unusual phenomenon which shares some of the opposite properties of fractals.

'How important is the concept of fractal in physics' has been witnessed by recent publications, conferences and meetings. The idea of fractals has greatly added to unifying the understanding of complex physical systems: for example, gelation and polymerisation in polymers, flocculation and coagulation in colloidal physics, percolation in a wide variety of systems (e.g. metal–insulator transitions, magnetic–nonmagnetic transition in dilute magnets etc), aggregation (diffusion limited, cluster–cluster, reaction limited etc) and dendretic growths in dust, soot and materials etc. Thanks are due to Mandelbrot (1982) who brought the mathematical concept closer to physical realities; he has popularised the subject (Dhar 1977) during the last few years to such an extent that the two words 'Mandelbrot' and 'fractal' have become synonymous—a classic example is his book 'The Fractal Geometry of Nature'. Apart from calculating just the fractal dimensionalities by studying the variation of the radius of gyration of ramified objects with the number of sites enclosed in it, there are several ways in which the notion of fractals provides physical insight into physical phenomena. The very simple study of random walk diffusion on homogeneous fractals (Leyvraz and Stanley 1983) has revealed many interesting results (Gefen *et al* 1983, Rammal and Toulouse 1983, Havlin and Ben-Avraham 1983, Pandey and Stauffer 1983, Pandey *et al* 1984) using the self-similarity arguments valid over the range of fractal dimensions. The superuniversal Alexander–Orbach (1982) conjecture valid for the homogeneous fractals came in a very elegant way into the framework of self similarities (the superuniversal law does not seem to be valid for the inhomogeneous fractals). Extrapolating the arguments based on these simple established results, here we discuss the possibility of a ramified system coupled with a walk which shares the opposite properties of a fractal.

Let us consider the random walk motion on a fractal of fractal dimensionality D_f embedded in a d-dimensional Euclidean space. The root mean square displacement R is given by (Rammal and Toulouse 1983)

$$R \sim t^k, \tag{1}$$

$$k = D_s/(2D_f), \tag{2}$$

where D_s is the spectral dimensionality of the fractal. On a Euclidean lattice where

$D_s = D_f$, the exponent k has the standard value $\frac{1}{2}$. On a fractal the exponent attains a value k_f such that $k_f \leq k$. For example, for the random walk motion on Sierpinsky gaskets,

$$\begin{aligned} k_f &= \ln 2/\ln 5 \qquad (D_f = \ln 3/\ln 2,\ d = 2), \\ &= \ln 2/\ln 6 \qquad (D_f = 2,\ d = 3). \end{aligned} \tag{3}$$

For the random walk motion on percolating fractals, we have (Gefen *et al* 1983, Pandey and Stauffer 1983, Pandey *et al* 1984, Havlin and Ben-Avraham 1983, Zabolitzky 1984)

$$\begin{aligned} k_f &\cong 0.33 \qquad (D_f \cong 1.8,\ d = 2), \\ &\cong 0.20 \qquad (D_f \cong 2.5,\ d = 3), \end{aligned} \tag{4}$$

at the percolation threshold where the infinite cluster shows self similarities at all length scales (the upper bound for the length scale of self similarity here is the percolation correlation length). There are several other examples of this kind, e.g. random walk on random walk (Ball and Cates 1984), random walk on DLA (Meakin and Stanley 1983) etc, which show similar exponent inequality. Thus, generalising this result, a walk of dimensionality $1/k$ in a Euclidean dimension d seems to have its dimensionality $1/k_f$ in an embedded fractal of dimensionality D_f such that

$$1/k_f > 1/k \qquad \text{for } D_f < d. \tag{5}$$

Now, if there exists a random system on which the walk has dimensionality $1/k_a$ such that

$$1/k_a < 1/k \tag{6}$$

then, in scientific jargon, it is tempting to call such random systems 'antifractal' in the context of our discussion here. Below we quote the possibility of such a random system.

Recently, Heinrichs and Kumar (1984) have discussed the motion of a particle subject to a continuously distributed random force with both static and dynamic components in a one-dimensional continuum. The model is defined by the equation of motion

$$x = F(x) + \eta(t) \tag{7}$$

where $F(x)$ and $\eta(t)$ are independently distributed random variables with gaussian white noise correlations,

$$\langle F(x)F(x')\rangle = F_0^2\delta(x-x'), \qquad \langle \eta(t)\eta(t')\rangle = \eta_0^2\delta(x-x'), \qquad \langle F(x)\rangle = \langle \eta(t)\rangle = 0. \tag{8}$$

They (Heinrichs and Kumar 1984) claim to show exactly that the average moments are given by (for all $m = 1, 2, \ldots$)

$$\langle x^{2m}(t)\rangle \sim t^{2m}, \qquad \langle x^{2m+1}\rangle = 0, \qquad t \to \infty, \tag{9}$$

and that this ballistic motion $\langle x^2(t)\rangle \sim t^2$ is induced by purely static disorder, independent of the time varying force component. This may be one example where the disorder is facilitating the random walk motion, changing from random walk to ballistic motion and, therefore, reducing the dimensionality of the walk (from 2 to 1) in contrast to the earlier examples of a random walk on fractals. This disorder medium coupled with a walk may then be said to exhibit an 'antifractal phenomenon'; this word is valid only in the context of properties mentioned here, and caution must be taken in using it in a general sense, as it may be misleading.

An immediate question arises, how to realise a random system which shows the unusual behaviour mentioned above. Let us try to model such space at least crudely. Consider a one-dimensional system with lattice sites distributed randomly (see figure 1). The lattice constant a has a distribution having values with minimum length a_{min}

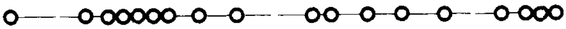

Figure 1. Schematic representation of one-dimensional Levy lattice.

to maximum length a_{max}. Let us consider a random walker executing its random walk with length step equal to a much larger than the minimum lattice constant; say it is equal to a_{max}. When the walker is in the rare region of space it is hopping on fewer sites than when it is in the compact region. The diffuser is always making jumps in forward and backward directions with the same length steps but it is passing by a different number of sites, say L; in the compact region, L is larger than in the rare region. If the distribution of the sites (or the occupied sites in a lattice) (see figure 1) is such that it causes the topological length step L of the random walker to satisfy the conditions

$$P(L > U) = U^{-f}, \qquad P(L < 1) = 0, \tag{10}$$

where $P(L > U)$ is the probability that the length step will be greater than or equal to U ($U > a_{max}$), then, topologically, the walker is making a Levy flight with varying length steps L which may give rise to walks of different dimensionalities lying between 1 and 2 for $1 < f < 2$. (Note that such a disordered system may be called a Levy lattice in analogy with a Levy flight.) A walk of an extreme dimensionality 1 (i.e. ballistic motion) can then be imagined without much conceptual difficulty. In any case, such a random medium which enhances the power law exponent of the walk from that of its value on the corresponding homogeneous lattice (the random walk value is $\frac{1}{2}$) would be said to exhibit an antifractal phenomenon. One may perhaps realise several geometries sharing these features including those in an embedded space of higher dimensions.

Thus our strange fractal system (which in a very crude sense may be called an antifractal system) shares the opposite properties of the fractals, namely, the value of the power law exponent for the random walk on a fractal is reduced with respect to its value on an embedded Euclidean space, while the value of this exponent for the random walk in the strange fractal system is enhanced with respect to its value on an embedded Euclidean space, i.e. the ratio of the topological and spectral dimensionality of the fractal system is greater than that of its value in a strange fractal system. Fractal and antifractal phenomena taking place in the same embedded space may annul the effect of each other, at least for the random walk behaviour considered above.

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